

On the maximum of a type of random processes

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Abstract

Let $\{\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}\}$ be a filtered probability space satisfying the usual conditions. We consider random processes X_t , $t \in [0, T]$, which satisfy the following condition:

$$\mathbb{E}(|\mathbb{E}(X_t|\mathcal{F}_s) - X_s|^p) \leq A_{p,h}|t - s|^{ph}, \quad \text{for all } 0 \leq s < t \leq T. \quad (1)$$

where $p > 1$ and $h \in (0, 1]$ are some constants satisfying $ph > 1$, and $A_{p,h}$ is a constant depending only on p and h . Typical examples of such processes are martingales and processes with the following increment control:

$$\mathbb{E}(|X_t - X_s|^p) \leq A_{p,h}|t - s|^{ph}, \quad \text{for all } s, t \in [0, T], \quad (2)$$

We are interested in estimate of the tail probability of the supremum

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq \lambda\right), \quad (3)$$

for which we will show that a Doob type inequality (see Theorem 1.1) holds for processes satisfying (1). As an application, we show that with the condition (2) given, the decay of (3) behaves (roughly speaking) in the same manner as the marginal

$$\mathbb{P}(|X_t| \geq \lambda).$$

1 A Doob type maximal inequality

Lemma 1.1. *For any $s_0, t_0 \in [0, T]$, $s_0 < t_0$, it holds that*

$$\mathbb{E}\left(\sup_{s_0 \leq s < t \leq t_0} |\mathbb{E}(X_t|\mathcal{F}_s) - X_s|^p\right) \leq C_{p,h,\theta} A_{p,h} |t_0 - s_0|^{ph},$$

where

$$C_{p,h,\theta} = [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1}\right)^p \left(\frac{4}{ph-1}\right)^{\theta(p-1)+1} \Gamma[\theta(p-1)+1],$$

with an arbitrary constant $\theta > 1$, $\zeta(\theta) = \sum_{m=1}^{\infty} m^{-\theta}$ is the Riemann zeta function and $\Gamma(z)$ is the Gamma function.

Proof. Let $s, t \in [s_0, t_0]$, $s < t$ be fixed temporarily. Denote by

$$I_l^m = [t_{l-1}^m, t_l^m] = s_0 + (t_0 - s_0) \times \left[\frac{l-1}{2^m}, \frac{l}{2^m}\right]$$

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the dyadic sub-intervals of $[s, t]$. Then there exists a sequence $\{J_k\} \subseteq \{I_l^m : 1 \leq l \leq 2^m, m \geq 0\}$ such that

- i) $J_k, k = 1, 2, \dots$, are mutually disjoint;
- ii) for any $m \geq 1$, there are at most two elements of $\{J_k\}$ with length $(t_0 - s_0)2^{-m}$;
- iii) $[s, t] = \cup_{k=1}^{\infty} J_k$.

Denote $J_k = [u_{k-1}, u_k]$. Then

$$\begin{aligned} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s| &= \left| \sum_{k=1}^{\infty} \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_s) \right| \\ &\leq \sum_{k=1}^{\infty} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \\ &= \sum_{m=1}^{\infty} \sum_{\{J_k : |J_k| = (t_0 - s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right], \end{aligned}$$

where $\Delta X_{J_k} = X_{u_k} - X_{u_{k-1}}$. Let $\xi_l^m = \mathbb{E}(\Delta X_{I_l^m} | \mathcal{F}_{t_{l-1}^m})$, $1 \leq l \leq 2^m$, $m = 1, 2, \dots$. For any $\theta > 1$, by Jensen's inequality,

$$\begin{aligned} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p &\leq \left(\sum_{m=1}^{\infty} \frac{1}{\zeta(\theta)m^\theta} \cdot \zeta(\theta)m^\theta \sum_{\{J_k : |J_k| = (t_0 - s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &\leq \sum_{m=0}^{\infty} \frac{1}{\zeta(\theta)m^\theta} \left(\zeta(\theta)m^\theta \sum_{\{J_k : |J_k| = (t_0 - s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &= \zeta(\theta)^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \left(\sum_{\{J_k : |J_k| = (t_0 - s_0)2^{-m}\}} \mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &\leq [2\zeta(\theta)]^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \sum_{\{J_k : |J_k| = (t_0 - s_0)2^{-m}\}} \left(\mathbb{E} \left[\left| \mathbb{E}(\Delta X_{J_k} | \mathcal{F}_{u_{k-1}}) \right| \middle| \mathcal{F}_s \right] \right)^p \\ &\leq [2\zeta(\theta)]^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \sup_{r \in [s_0, t_0]} [\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p, \end{aligned}$$

where the inequality in the fourth line is due to the property ii) of $\{J_k\}$. Hence,

$$\sup_{s_0 \leq s < t \leq t_0} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p \leq [2\zeta(\theta)]^{p-1} \sum_{m=0}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \sup_{r \in [s_0, t_0]} [\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p.$$

By Doob's maximal inequality for martingales,

$$\begin{aligned}
\mathbb{E} \left(\sup_{s_0 \leq s < t \leq t_0} |\mathbb{E}(X_t | \mathcal{F}_s) - X_s|^p \right) &\leq [2\zeta(\theta)]^{p-1} \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \mathbb{E} \left(\sup_{r \in [s_0, t_0]} [\mathbb{E}(|\xi_l^m| | \mathcal{F}_r)]^p \right) \\
&\leq [2\zeta(\theta)]^{p-1} \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \left(\frac{p}{p-1} \right)^p \mathbb{E} ([\mathbb{E}(|\xi_l^m| | \mathcal{F}_{t_0})]^p) \\
&\leq [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} \sum_{l=1}^{2^m} \mathbb{E} (|\xi_l^m|^p) \\
&\leq A_{p,h} [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1} \right)^p \sum_{m=1}^{\infty} m^{\theta(p-1)} \cdot 2^m \cdot \left(\frac{|t_0 - s_0|}{2^m} \right)^{ph} \\
&= C_{p,h,\theta} A_{p,h} |t_0 - s_0|^{ph},
\end{aligned}$$

where $C_{p,h,\theta} = [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1} \right)^p [\sum_{m=1}^{\infty} m^{\theta(p-1)} \cdot 2^{-m(ph-1)}]$. Note that

$$\begin{aligned}
\sum_{m=1}^{\infty} m^{\theta(p-1)} \cdot 2^{-m(ph-1)} &\leq \sum_{m=1}^{\infty} 2^{\theta(p-1)} \int_{m-1}^m r^{\theta(p-1)} e^{-r(ph-1) \log 2} dr \\
&\leq \left(\frac{4}{ph-1} \right)^{\theta(p-1)+1} \int_0^{\infty} r^{\theta(p-1)} e^{-r} dr \\
&= \left(\frac{4}{ph-1} \right)^{\theta(p-1)+1} \Gamma[\theta(p-1)+1].
\end{aligned}$$

This completes the proof. \square

As an application of Lemma 1.1, we show that a Doob-type inequality holds for processes satisfying the condition (1). To this end, we shall need the following elementary result.

Lemma 1.2. *Let Y_t , $t \in [0, T]$, be any right continuous random process such that Y_t is integrable for each t , and let $0 \leq s_0 < t_0 \leq T$. Then*

1) *For any stopping time τ with $s_0 \leq \tau \leq t_0$, it holds that*

$$|\mathbb{E}(Y_{t_0} | \mathcal{F}_\tau) - Y_\tau| \leq \mathbb{E} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| \middle| \mathcal{F}_\tau \right]. \quad (4)$$

2) *For any $\lambda > 0$, it holds that*

$$\mathbb{P} \left(\sup_{u \in [s_0, t_0]} Y_u \geq \lambda \right) \leq \frac{1}{\lambda} \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0} | \mathcal{F}_u) - Y_u| + Y_{t_0} \right] d\mathbb{P}. \quad (5)$$

Proof. 1) By the right continuity of Y_t , we may assume that τ takes only countably many values $\{u_k :$

$k = 1, 2, \dots\} \subseteq [s_0, t_0]$. Then

$$\begin{aligned}
|\mathbb{E}(Y_{t_0}|\mathcal{F}_\tau) - Y_\tau| &= \sum_{k=1}^{\infty} |\mathbb{E}(Y_{t_0}|\mathcal{F}_\tau) - Y_\tau| 1_{\{\tau=u_k\}} \\
&= \sum_{k=1}^{\infty} \left| \mathbb{E} \left[(Y_{t_0} - Y_\tau) 1_{\{\tau=u_k\}} \middle| \sigma(\mathcal{F}_\tau \cap \{\tau=u_k\}) \right] \right| \\
&= \sum_{k=1}^{\infty} \left| \mathbb{E} \left[\mathbb{E} \left((Y_{t_0} - Y_{u_k}) \middle| \mathcal{F}_{u_k} \right) 1_{\{\tau=u_k\}} \middle| \sigma(\mathcal{F}_\tau \cap \{\tau=u_k\}) \right] \right| \\
&\leq \sum_{k=1}^{\infty} \mathbb{E} \left[\left(\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0}|\mathcal{F}_u) - Y_u| \right) 1_{\{\tau=u_k\}} \middle| \sigma(\mathcal{F}_\tau \cap \{\tau=u_k\}) \right] \\
&= \sum_{k=1}^{\infty} \mathbb{E} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0}|\mathcal{F}_u) - Y_u| \middle| \mathcal{F}_\tau \right] 1_{\{\tau=u_k\}} \\
&= \mathbb{E} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0}|\mathcal{F}_u) - Y_u| \middle| \mathcal{F}_\tau \right].
\end{aligned}$$

2) Let $\tau = \inf \{u \in [s_0, t_0] : Y_u \geq \lambda\} \wedge T$. Then $\left\{ \sup_{t \in [s_0, t_0]} Y_u \geq \lambda \right\} = \{\tau < T\} \cup \{\tau = t_0, Y_{t_0} \geq \lambda\} \in \mathcal{F}_\tau$. Therefore, by (4),

$$\begin{aligned}
\int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} Y_\tau d\mathbb{P} &= - \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} (\mathbb{E}(Y_{t_0}|\mathcal{F}_\tau) - Y_\tau) d\mathbb{P} + \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} Y_{t_0} d\mathbb{P} \\
&\leq \int_{\{\sup_{u \in [s_0, t_0]} Y_u \geq \lambda\}} \left[\sup_{u \in [s_0, t_0]} |\mathbb{E}(Y_{t_0}|\mathcal{F}_u) - Y_u| + Y_{t_0} \right] d\mathbb{P}.
\end{aligned}$$

□

Proposition 1.1. Let $0 \leq s_0 < t_0 \leq T$, and let $X^* = \sup_{u \in [s_0, t_0]} |X_u|$. Then for any $1 < q \leq p$,

$$\|X^*\|_{L^q} \leq \frac{q}{q-1} \left[C_{p,h,\theta}^{1/p} A_{p,h}^{1/p} |t_0 - s_0|^h + \|X_{t_0}\|_{L^q} \right]. \quad (6)$$

where $C_{p,h,\theta}$ is a constant which differs from the constant $C_{p,h,\theta}$ in Lemma 1.1 by a multiple depending only on p , and $\delta > 0$ is an arbitrary constant.

Proof. Denote $Y = \sup_{u \in [s_0, t_0]} |\mathbb{E}(X_{t_0}|\mathcal{F}_u) - X_u| + |X_{t_0}|$. Then $\{X^* \geq \lambda\} \subseteq \left\{ \sup_{u \in [s_0, t_0]} X_u \geq \lambda \right\}$. By Lemma 1.2.2 and Lemma 1.1

$$\begin{aligned}
\|X^*\|_{L^q}^q &= q \int_0^\infty \lambda^{q-1} \mathbb{P}(X^* \geq \lambda) d\lambda \\
&\leq q \int_0^\infty \lambda^{q-2} \int_{\{\sup_{u \in [s_0, t_0]} X_u \geq \lambda\}} Y d\mathbb{P} d\lambda \\
&\leq q \int_0^\infty \lambda^{q-2} \int_{\{X^* \geq \lambda\}} Y d\mathbb{P} d\lambda \\
&= q \int_\Omega \left(\int_0^{X^*} \lambda^{q-2} d\lambda \right) Y d\mathbb{P} \\
&= \frac{q}{q-1} \int_\Omega |X^*|^{q-1} Y d\mathbb{P} \\
&\leq \frac{q}{q-1} \|X^*\|_{L^q}^{q/q'} \|Y\|_{L^q},
\end{aligned}$$

where q' is the conjugate exponent of q . Therefore,

$$\begin{aligned}\|X^*\|_{L^q} &\leq \frac{q}{q-1} \|Y\|_{L^q} \\ &\leq \frac{q}{q-1} \left(\left\| \sup_{u \in [s_0, t_0]} |\mathbb{E}(X_{t_0} | \mathcal{F}_u) - X_u| \right\|_{L^q} + \|X_{t_0}\|_{L^q} \right) \\ &\leq \frac{q}{q-1} \left[C_{p,h,\theta}^{1/p} A_{p,h}^{1/p} |t_0 - s_0|^h + \|X_{t_0}\|_{L^q} \right].\end{aligned}$$

□

2 Tail decay of the supremum

Definition 2.1. The marginals of the process X_t are said to have *uniform α -exponential decay*, if there exist constants $\alpha > 0$, $C > 0$ and $D > 0$ such that

$$\mathbb{P}(|X_t| \geq \lambda) \leq C \exp(-D\lambda^\alpha), \quad \text{for all } \lambda > 0 \text{ and all } t \in [0, T]. \quad (7)$$

We shall show that the distributions of the $\sup_{t \in [0, T]} |X_t|$ has α -exponential decay, if and only if the marginals of X_t have uniform α -exponential decay. It follows from a simple computation that

Lemma 2.1. *Let X_t , $t \in [0, T]$ be a random process satisfying (1), and let $q > 0$. Then*

$$\mathbb{E}(|X_t|^q) \leq CD^{-q/\alpha} \Gamma\left(\frac{q}{\alpha} + 1\right).$$

Theorem 2.1. *Let X_t , $t \in [0, T]$ be a random process satisfying (1). Suppose that there exist constants $\alpha > 0$, $D > 0$, and $\delta_0 \geq 0$ such that*

$$\mathbb{P}(|X_t| \geq \lambda) \leq C \exp(-D\lambda^\alpha), \quad \text{for all } \lambda > 0 \text{ and all } t \in [0, T].$$

Then

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda\right) \leq K\lambda^{-\frac{1}{\alpha}} \exp\left[-\left(1 - \frac{1}{ph}\right) D\lambda^\alpha\right], \quad \text{for all } \lambda \geq \delta_0,$$

where $K = 4T \left[C_{p,h,\theta} A_{p,h}\right]^{\frac{1}{ph}} \left[1 + C \left(\frac{p}{p-1}\right)^p\right]^{1-\frac{1}{ph}}$, and the constant $C_{p,h,\theta}$ is the same as in Lemma 1.1.

Proof. For $N \in \mathbb{N}_+$, let $I_n = [t_{n-1}, t_n] = [(n-1)T/N, nT/N]$. Then

$$\left\{ \sup_{t \in [0, T]} |X_t| \geq 2\lambda \right\} \subseteq \bigcup_{n=1}^N \left\{ \sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t) - X_t| \geq \lambda \right\} \bigcup \bigcup_{n=0}^{N-1} \left\{ \sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda \right\}.$$

Therefore,

$$\mathbb{P}\left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda\right) \leq \sum_{n=1}^N \mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t) - X_t| \geq \lambda\right) + \sum_{n=0}^{N-1} \mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda\right). \quad (8)$$

By Lemma 1.1,

$$\mathbb{P}\left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t) - X_t| \geq \lambda\right) \leq C_{p,h,\theta} A_{p,h} \frac{1}{\lambda^p} \left(\frac{T}{N}\right)^{ph}. \quad (9)$$

We need to estimate $\mathbb{P}(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda)$. If $\alpha > p$, by Doob's inequality and Lemma 2.1,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^\alpha \right) &\leq \left(\frac{\alpha}{\alpha - 1} \right)^\alpha \mathbb{E}(|X_{t_n}|^\alpha) \\ &\leq C \left(\frac{p}{p - 1} \right)^p D^{-1}. \end{aligned}$$

If $\alpha \leq p$, the above yields that

$$\mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^\alpha \right) \leq \mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^p \right)^{\alpha/p} \leq C \left(\frac{p}{p - 1} \right)^p D^{-1}.$$

Moreover, for any $q \geq 2$, by a similar argument,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^{\alpha q} \right) &\leq \mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(|X_{t_n}|^\alpha | \mathcal{F}_t)|^q \right) \\ &\leq \left(\frac{q}{q - 1} \right)^q \mathbb{E}(|X_{t_n}|^{\alpha q}) \\ &\leq C \left(\frac{q}{D(q - 1)} \right)^q \Gamma(q + 1) \\ &\leq C(2D^{-1})^q \Gamma(q + 1). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{D}{4} \sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^\alpha \right) \right] &= \sum_{q=0}^{\infty} \frac{(D/4)^q}{q!} \mathbb{E} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)|^{\alpha q} \right) \\ &\leq 1 + \frac{C}{4} \left(\frac{p}{p - 1} \right)^p + C \sum_{q=2}^{\infty} 2^{-q} \\ &\leq 2 \left[1 + C \left(\frac{p}{p - 1} \right)^p \right]. \end{aligned}$$

By Chebyshev's inequality,

$$\mathbb{P} \left(\sup_{t \in I_n} |\mathbb{E}(X_{t_n} | \mathcal{F}_t)| \geq \lambda \right) \leq 2 \left[1 + C \left(\frac{p}{p - 1} \right)^p \right] \exp \left(-\frac{D}{4} \lambda^\alpha \right). \quad (10)$$

Therefore, by (8), (9) and (10),

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda \right) \leq C_{p,h,\theta} A_{p,h} \frac{N}{\lambda^p} \left(\frac{T}{N} \right)^{ph} + 2N \left[1 + C \left(\frac{p}{p - 1} \right)^p \right] \exp \left(-\frac{D}{4} \lambda^\alpha \right).$$

Setting N to be the integer part of $\left[C_{p,h,\theta} A_{p,h} T^{ph} \lambda^{-p} \exp(D\lambda^\alpha) \right]^{\frac{1}{ph}}$ gives that

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t| \geq 2\lambda \right) \leq 4T \left[C_{p,h,\theta} A_{p,h} \right]^{\frac{1}{ph}} \left[1 + C \left(\frac{p}{p - 1} \right)^p \right]^{p \left(1 - \frac{1}{ph} \right)} \lambda^{-\frac{1}{h}} \exp \left[-\left(1 - \frac{1}{ph} \right) D \lambda^\alpha \right].$$

□

Example 2.1. We consider the tail decay of the supremum of a standard fractional Brownian motion B_t^h , $t \in [0, T]$, with Hurst parameter $h \in (0, 1)$, that is, a Gaussian process with $B_0^h = 0$ and covariance

function

$$R(t, s) = \frac{1}{2} (|t|^{2h} + |s|^{2h} - |t - s|^{2h}), \quad t, s \in [0, T].$$

For the fractional Brownian motion, one has $B_t^h - B_s^h \sim N(0, \frac{1}{2}|t - s|^{2h})$, and therefore,

$$\mathbb{E}(|B_t^h - B_s^h|^p) = A_p |t - s|^{ph}, \quad t, s \in [0, T],$$

where

$$A_p = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right).$$

For any $t \in [0, 1]$ and any $\lambda > 0$, one has

$$\begin{aligned} \mathbb{P}(|B_t^h| \geq \lambda) &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\lambda/t^h}}^{\infty} \exp\left(-\frac{1}{2}u^2\right) du \\ &\leq \frac{t^h}{2\sqrt{\pi}\lambda} \exp\left(-\frac{\lambda^2}{t^{2h}}\right) \\ &\leq \frac{1}{2\sqrt{\pi}\lambda} \exp(-\lambda^2). \end{aligned}$$

Now put $\phi(\lambda) = \frac{1}{2\sqrt{\pi}\lambda} \exp(-\lambda^2)$. For any $p > 1/h$, by Lemma 2.1.2,

$$\mathbb{P}\left(\sup_{t \in [0, 1]} |B_t^h| \geq 2\lambda\right) \leq 2 [C_{p,h,\theta} A_p]^{\frac{1}{ph}} \lambda^{-\frac{1}{h}} \phi(\lambda)^{1-\frac{1}{ph}}, \quad \text{for all } \lambda > 0,$$

where

$$C_{p,h,\theta} = [2\zeta(\theta)]^{p-1} \left(\frac{p}{p-1}\right)^p \left(\frac{4}{ph-1}\right)^{\theta(p-1)+1} \Gamma[\theta(p-1)+1]$$

with an arbitrary constant $\theta > 1$. Setting $p = 2/h$, $\theta = p/(p-1)$ gives

$$\mathbb{P}\left(\sup_{t \in [0, 1]} |B_t^h| \geq 2\lambda\right) \leq \frac{C_h}{\lambda} \exp\left(-\frac{\lambda^2}{2}\right), \quad \text{for all } \lambda > 0,$$

where C_h is a constant depending only on h . By scaling, we deduce that

$$\mathbb{P}\left(\sup_{t \in [0, T]} |B_t^h| \geq 2\lambda\right) \leq \frac{C_h T^h}{\lambda} \exp\left(-\frac{\lambda^2}{2T^{2h}}\right), \quad \text{for all } \lambda > 0.$$

Example 2.2. Let $a = (a_1, a_2, \dots)$ be a sequence of real numbers, let $f_k(t)$, $k = 1, 2, \dots$, be a sequence of real functions defined on $[0, T]$, and let ξ_k , $k = 1, 2, \dots$, be i.i.d. Rademacher random variables. In [?], Theorem I, p. 339, R. Paley and A. Zygmund showed that if

$$\sum_{k=1}^{\infty} a_k^2 < \infty$$

and

$$\int_0^T f_k(t)^2 dt \leq A, \quad k = 1, 2, \dots,$$

for some constant $A < \infty$, then for almost all $\omega \in \Omega$, the series $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$ converges for a.e. $t \in [0, T]$, and the limit is an element in $L^2([0, T])$.

Let $H \subseteq [0, T] \times \Omega$ be the set of (t, ω) at which $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$ converges. Then the projection

of H on Ω has probability one. We shall show that, under some stronger assumption, $\sum_{k=1}^{\infty} a_k \xi_k f_k(t)$ converges uniformly in $t \in H^\omega$ for any $\omega \in \Omega$. Here $H^\omega = \{t \in [0, T] : (t, \omega) \in H\}$ has Lebesgue measure T for almost all $\omega \in \Omega$. Suppose that $h \in (0, 1)$ and that $f_k(t)$, $k = 1, 2, \dots$, be h -Hölder continuous with Hölder constants L_k , that is,

$$|f_k(t) - f_k(s)| \leq L_k |t - s|^h.$$

Suppose that

$$\sum_{k=1}^{\infty} a_k^2 [f_k(0)^2 + L_k^2] < \infty.$$

Then for a.e. $\omega \in \Omega$,

$$\sum_{k=1}^{\infty} a_k \xi_k f_k(t)$$

converges uniformly in $t \in [0, T]$.

Proof. Let $X(t, \omega) = \sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t) 1_H(t, \omega)$. And to simplify the notation, we refer to $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t) 1_H(t, \omega)$ by simply writing $\sum_{k=1}^{\infty} a_k \xi_k(\omega) f_k(t)$.

For any $p > 0$, by Khintchine's inequality,

$$\mathbb{E} \left[\left| \sum_{k=1}^{\infty} a_k \xi_k f_k(0) \right|^p \right] \leq C_p \left[\sum_{k=1}^{\infty} a_k^2 f_k(0)^2 \right]^{p/2},$$

where

$$C_p = \max \left(\sqrt{\frac{2^p}{\pi}} \Gamma \left(\frac{p+1}{2} \right), 1 \right).$$

Similarly,

$$\mathbb{E} \left[\left| \sum_{k=1}^{\infty} a_k \xi_k (f_k(t) - f_k(0)) \right|^p \right] \leq C_p \left[\sum_{k=1}^{\infty} a_k^2 L_k^2 \right]^{p/2} t^{ph}.$$

By the above, we see that

$$\mathbb{E} (|X_t|^p) \leq 2^{p/2} C_p \left[\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \quad (11)$$

and

$$\mathbb{E} (|X_t - X_s|^p) \leq 2^{p/2} C_p \left[\sum_{k=1}^{\infty} a_k^2 L_k^2 \right]^{p/2} |t - s|^{ph} \quad (12)$$

for all $p > 0$. Put

$$A_{p,h} = \frac{2^p}{\sqrt{\pi}} \Gamma \left(\frac{p+1}{2} \right) \left[\sum_{k=1}^{\infty} a_k^2 L_k^2 \right]^{p/2}. \quad (13)$$

Then condition (2) is satisfied for any $p > 1/h$.

We now give an estimate of the tail decay of the marginals X_t . For any $u \geq 0$, by (11) and Stirling's

formula,

$$\begin{aligned}
\mathbb{E}[\exp(u|X_t|)] &\leq \sum_{p=0}^{\infty} \frac{2^{p/2} u^p C_p}{p!} \left[\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \\
&\leq K \sum_{p=0}^{\infty} 2^p \frac{\Gamma(\frac{p+1}{2})}{\Gamma(p+1)} \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \\
&= K \sum_{p=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(\frac{p}{2} + 1)} \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{p/2} \\
&\leq K \left(\sum_{p=0}^{\infty} \frac{1}{\Gamma(\frac{p}{2} + 1)^2} \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^p \right)^{1/2}
\end{aligned}$$

where K is a universal constant that might be different from line to line. Since

$$\Gamma\left(\frac{p}{2} + 1\right)^2 \geq \Gamma\left(\frac{p+2}{2}\right) \Gamma\left(\frac{p+1}{2}\right) = 2^{-p} \sqrt{\pi} \Gamma(p+1) = \frac{\sqrt{\pi} p!}{2^p},$$

we obtain that

$$\mathbb{E}[\exp(u|X_t|)] \leq K \left(\sum_{p=0}^{\infty} \frac{1}{p!} \left[2u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^p \right)^{1/2} = K \exp \left[u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right].$$

Now, by Chebyshev's inequality,

$$\mathbb{P}(|X_t| \geq \lambda) \leq K \exp \left[-u\lambda + u^2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right].$$

Setting $u = \lambda \left[2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \right]^{-1}$ gives that

$$\mathbb{P}(|X_t| \geq \lambda) \leq K \exp \left[-\frac{\lambda^2}{2 \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2)} \right]. \quad (14)$$

Now by Theorem 2.1, we have

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left| \sum_{k=1}^{\infty} a_k \xi_k f_k(t) \right| \geq 2\lambda \right) \leq C_h \exp \left[-\frac{D_h \lambda^2}{\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2)} \right], \quad (15)$$

where C_h, D_h are constants depending only on $h \in (0, 1)$.

Now, for any $u \in [0, 1]$, denote $\sigma^2 = \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2)$, and let

$$l(u) = \inf \left\{ l \in \mathbb{R}_+ : \int_0^l \sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) 1_{(k-1, k]}(s) ds > u\sigma^2 \right\}. \quad (16)$$

Then $l(u) \rightarrow \infty$ as $u \rightarrow 1$.

To show the a.s. uniform convergence of $\sum_{k=1}^{\infty} a_k \xi_k f_k(t)$, we need to show that

$$\mathbb{P} \left(\bigcap_{0 < u < 1} \left\{ \sup_{n \geq l(u)} \sup_{t \in [0, T]} \left| \sum_{k=n}^{\infty} a_k \xi_k f_k(t) \right| \geq 2\lambda \right\} \right) = 0$$

for any $\lambda > 0$. Clearly, it suffices to show that

$$\lim_{u \rightarrow 1} \mathbb{P} \left(\sup_{n \geq l(u)} \sup_{t \in [0, T]} \left| \sum_{k=n}^{\infty} a_k \xi_k f_k(t) \right| \geq 2\lambda \right) = 0. \quad (17)$$

Define

$$Y_u = \sup_{t \in [0, T]} \left| \int_0^{l(u)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|, \quad u \in [0, 1].$$

To prove (17), it suffices to show that

$$\lim_{u \rightarrow 1} \mathbb{P} \left(\sup_{v \in [u, 1]} Y_v \geq 2\lambda \right) = 0. \quad (18)$$

We first note that, for $0 \leq u < v \leq 1$,

$$|Y_v - Y_u| \leq \sup_{t \in [0, T]} \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|.$$

In fact, let $t^* \in \arg \max_{t \in [0, T]} \left| \int_0^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|$. Then

$$\begin{aligned} Y_v - Y_u &\leq \left| \int_0^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t^*) 1_{(k-1, k]}(s) ds \right| - \left| \int_0^{l(u)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t^*) 1_{(k-1, k]}(s) ds \right| \\ &\leq \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t^*) 1_{(k-1, k]}(s) ds \right| \\ &\leq \sup_{t \in [0, T]} \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|. \end{aligned} \quad (19)$$

Similarly, $Y_u - Y_v \leq \sup_{t \in [0, T]} \left| \int_{l(u)}^{l(v)} \sum_{k=1}^{\infty} a_k \xi_k f_k(t) 1_{(k-1, k]}(s) ds \right|$.

For any $u < v$, by the definition of $l(u)$,

$$\sum_{k=1}^{\infty} a_k^2 (f_k(0)^2 + L_k^2) \int_{l(u)}^{l(v)} 1_{(k-1, k]}(s) ds = (v - u) \sigma^2. \quad (20)$$

Now, applying (15) to the sequence (a'_1, a'_2, \dots) with

$$a'_k = a_k \left(\int_{l(u)}^{l(v)} 1_{(k-1, k]}(s) ds \right)^{1/2}, \quad k \geq 1,$$

we obtain that

$$\mathbb{P}(|Y_v - Y_u| \geq 2\lambda) \leq C_h \exp \left[-\frac{D_h \lambda^2}{|v - u| \sigma^2} \right], \quad (21)$$

where C_h and D_h are constants depending only on h and might vary from line to line. In particular, noting that $Y_1 = 0$,

$$\mathbb{P}(|Y_v| \geq 2\lambda) \leq C_h \exp \left[-\frac{D_h \lambda^2}{(1 - u) \sigma^2} \right], \quad \text{for all } v \in [u, 1]. \quad (22)$$

Since (21) implies that

$$\mathbb{E}(|Y_v - Y_u|^p) \leq C_h |v - u|^{p/2}$$

for any $p > 2$. We are now in a position to apply Theorem 2.1 again, and deduce that

$$\mathbb{P} \left(\sup_{v \in [u, 1]} Y_v \geq 2\lambda \right) \leq C_h \exp \left[-\frac{D_h \lambda^2}{(1-u)\sigma^2} \right].$$

Thus, (18) follows readily. \square

3 An estimate for the up-crossing number of processes with increment controls

We now give an estimate for the up-crossing number of processes X_t which satisfies the condition (2).

Lemma 3.1. *For any $0 < q \leq p$, $0 < \alpha < \frac{h-1/p}{1/q-1/p}$, and any random times τ, σ such that $0 \leq \sigma \leq \tau \leq T$, it holds that*

$$\mathbb{E}(|X_\tau - X_\sigma|^q) \leq K_{q,\alpha} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^{qh} \mathbb{E} \left(\left| \frac{\tau - \sigma}{T} \right|^\alpha \right)^{1-q/p}, \quad (23)$$

where $K_{q,\alpha} = 4^q [1 - 2^{-q(h-1/p)+(1-q/p)\alpha}]^{-1}$, and the constant $C_{p,h,\theta}$ is the same as in Lemma 1.1.

Remark. It is noticed that, when $q = p$, $K_{q,\alpha} = 4^p [1 - 2^{1-ph}]^{-1}$ for all $\alpha > 0$.

Proof. We first note that, by virtue of Lemma 1.1.1 applied to the filtration $\mathcal{F}_t = \mathcal{F}_T$, $0 \leq t \leq T$, one has

$$\mathbb{E}(|X_\tau - X_\sigma|^p) \leq 2^{p-1} C_{p,h,\theta} A_{p,h} |t_0 - s_0|^{ph} \quad (24)$$

for any random times τ and σ with $s_0 \leq \sigma \leq \tau \leq t_0$.

Now, for any random times τ, σ with $0 \leq \sigma \leq \tau \leq T$, define

$$A_{r,k} = \{T(r-1)2^{-k} \leq \sigma < Tr2^{-k} < T(r+1)2^{-k} \leq \tau < T(r+2)2^{-k}\}, \quad 1 \leq r \leq 2^k - 1, k \geq 1.$$

Then $\{\tau \neq \sigma\} = \bigcup_{r,k} A_{r,k}$, and the union of sets is disjoint. Therefore,

$$X_\tau - X_\sigma = \sum_{r,k} (X_\tau - X_\sigma) 1_{A_{r,k}}.$$

Let

$$\begin{aligned} \tau_{r,k} &= \left(\tau \vee \frac{Tr}{2^k} \right) \wedge \frac{T(r+1)}{2^k}, \\ \sigma_{r,k} &= \left(\sigma \vee \frac{Tr}{2^k} \right) \wedge \frac{T(r+1)}{2^k}. \end{aligned}$$

Then

$$X_\tau - X_\sigma = \sum_{r,k} (X_{\tau_{r+1,k}} - X_{\sigma_{r-1,k}}) 1_{A_{r,k}}.$$

Since $A_{r,k}$ are mutually disjoint, we have

$$\begin{aligned}\mathbb{E}(|X_\tau - X_\sigma|^q) &= \mathbb{E}\left(\left|\sum_{r,k} (X_{\tau_{r+1,k}} - X_{\sigma_{r-1,k}}) 1_{A_{r,k}}\right|^q\right) \\ &= \mathbb{E}\left(\sum_{r,k} |X_{\tau_{r+1,k}} - X_{\sigma_{r-1,k}}|^q 1_{A_{r,k}}\right) \\ &\leq \sum_{r,k} [\mathbb{E}(|X_{\tau_{r+1,k}} - X_{\sigma_{r-1,k}}|^p)]^{q/p} \mathbb{P}(A_{r,k})^{1-q/p}.\end{aligned}$$

Note that $T(r-1)2^{-k} \leq \sigma_{r-1,k} \leq \tau_{r+1,k} < T(r+2)2^{-k}$. By (24),

$$\begin{aligned}\mathbb{E}(|X_\tau - X_\sigma|^q) &\leq \sum_{r,k} \left(2^{p-1} C_{p,h,\theta} A_{p,h} \left(\frac{T}{2^{k-2}}\right)^{ph}\right)^{q/p} \mathbb{P}(A_{r,k})^{1-q/p} \\ &\leq 2^q C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} \sum_{r,k} \left(\frac{T}{2^{k-2}}\right)^{qh} \mathbb{P}(A_{r,k})^{1-q/p}.\end{aligned}\tag{25}$$

By the fact that $\bigcup_{r=1}^{2^k-2} A_{r,k} \subseteq \{|\tau - \sigma| > T2^{-k}\}$ and Chebyshev's inequality,

$$\sum_{r=1}^{2^k-1} \mathbb{P}(A_{r,k}) \leq \mathbb{P}(|\tau - \sigma| > T2^{-k}) \leq 2^{k\alpha} \mathbb{E}\left(\left|\frac{\tau - \sigma}{T}\right|^\alpha\right).$$

By Jensen's inequality,

$$\begin{aligned}\sum_{r=1}^{2^k-1} \mathbb{P}(A_{r,k})^{1-q/p} &\leq 2^k \left[2^{-k} \sum_{r=1}^{2^k-1} \mathbb{P}(A_{r,k})\right]^{1-q/p} \\ &\leq 2^{k[q/p+(1-q/p)\alpha]} \left[\mathbb{E}\left(\left|\frac{\tau - \sigma}{T}\right|^\alpha\right)\right]^{1-q/p}.\end{aligned}\tag{26}$$

Therefore, by (25) and (26),

$$\begin{aligned}\mathbb{E}(|X_\tau - X_\sigma|^q) &\leq 2^q C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} \sum_{k=1}^{\infty} \left(\frac{T}{2^{k-2}}\right)^{qh} \sum_{r=1}^{2^k-1} \mathbb{P}(A_{r,k})^{1-q/p} \\ &\leq 4^q C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^{qh} \sum_{k=1}^{\infty} 2^{k[q/p+(1-q/p)\alpha-qh]} \left[\mathbb{E}\left(\left|\frac{\tau - \sigma}{T}\right|^\alpha\right)\right]^{1-q/p}.\end{aligned}$$

Since $\alpha < \frac{h-1/p}{1/q-1/p}$, setting $K_{q,\alpha} = 4^q \sum_{k=2}^{\infty} 2^{k[q/p+(1-q/p)\alpha-qh]} < 4^q [1 - 2^{-q(h-1/p)+(1-q/p)\alpha}]^{-1}$ completes the proof. \square

Definition 3.1. $\{Y_t : t \in [0, T]\}$ be a random process. Let $D : 0 = t_0 < \dots < t_N = T$ be a finite subset of $[0, T]$. For any $a, b \in \mathbb{R}$, $a < b$, let

$$T_0 = \inf\{t \in D : Y_t < a\}, T_1 = \inf\{t \in D : t > T_0, Y_t > b\},$$

$$T_{2k} = \inf\{t \in D : t > T_{2k-1}, Y_t < a\}, T_{2k+1} = \inf\{t \in D : t > T_{2k}, Y_t > b\}, \quad k \geq 1.$$

The up-crossing number $U_a^b(D)$ of Y_t through $[a, b]$ sampled in D is given by

$$U_a^b(D) = \sup \{k \geq 1 : T_{2k-1} \leq T\}.$$

And the up-crossing number U_a^b of Y_t through $[a, b]$ is defined as

$$U_a^b = \sup_D U_a^b(D),$$

where \sup_D is taken over all finite subsets D of any countable dense subset of $[0, T]$.

By definition, one has

$$\{U_a^b(D) \geq k\} = \{T_{2k-1} \leq T\}, \quad k \geq 1$$

and

$$\{U_a^b(D) = k\} = \{T_{2k-1} \leq T, T_{2k+1} = \infty\}, \quad k \geq 1.$$

For the up-crossing number $U_a^b(D)$ of a general random process Y_t , $t \in [0, T]$, we have the following elementary but useful result.

Lemma 3.2. *With the same notation as in Definition 3.1, one has*

$$(b-a)1_{\{U_a^b(D) \geq k\}} \leq -(Y_T - Y_{T_{2(k-1)}})1_{\{T_{2(k-1)} \leq T, T_{2k-1} = \infty\}} + Y_{T_{2k-1} \wedge T} - Y_{T_{2(k-1)} \wedge T},$$

for any $k \geq 1$.

Proposition 3.1. *Let $D : 0 = t_0 < \dots < t_N = T$ be a finite subset of $[0, T]$, and let $U_a^b(D)$ be the up-crossing number of X_t through $[a, b]$ sampled in D . Then, for any $0 < \delta < 1 - \frac{1}{ph}$,*

$$\mathbb{E}(U_a^b(D)^\delta) < \frac{K_\delta}{b-a} T^h,$$

where

$$K_\delta = 2 \left(C_{p,h,\theta}^{1/p} A_{p,h}^{1/p} + 4^q \left[1 - 2^{-q(h-1/p)+(1-q/p)\alpha} \right]^{-1} \zeta \left(\frac{1-\delta}{1-\alpha(1-q/p)} \right)^{1-\alpha(1-q/p)} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} \right),$$

and q and α are any constants satisfying $\frac{\delta}{h-1/p} < q < 1/h$ and $\frac{\delta}{1-q/p} < \alpha < \frac{h-1/p}{1/q-1/p}$.

Proof. By Lemma 3.2 and Lemma 3.1,

$$\begin{aligned} \frac{b-a}{k^{1-\delta}} \mathbb{P}(U_a^b(D) \geq k) &\leq \frac{1}{k^{1-\delta}} \mathbb{E} \left[\sup_{u,v \in [0,T]} |X_u - X_v| 1_{\{T_{2(k-1)} \leq T, T_{2k-1} = \infty\}} \right] + \frac{1}{k^{1-\delta}} \mathbb{E}(Y_{T_{2k-1} \wedge T} - Y_{T_{2(k-1)} \wedge T}) \\ &\leq \mathbb{E} \left[\sup_{u,v \in [0,T]} |X_u - X_v| 1_{\{T_{2(k-1)} \leq T, T_{2k-1} = \infty\}} \right] + \frac{1}{k^{1-\delta}} \mathbb{E}(|Y_{T_{2k-1} \wedge T} - Y_{T_{2(k-1)} \wedge T}|) \\ &\leq \mathbb{E} \left[\sup_{u,v \in [0,T]} |X_u - X_v| 1_{\{T_{2(k-1)} \leq T, T_{2k-1} = \infty\}} \right] \\ &\quad + \frac{1}{k^{1-\delta}} K_{q,\alpha} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^h \mathbb{E} \left(\left| \frac{T_{2k-1} \wedge T - T_{2(k-1)} \wedge T}{T} \right|^\alpha \right)^{1-q/p}, \end{aligned}$$

where $0 < q \leq p$, $0 < \alpha < \frac{h-1/p}{1/q-1/p}$, and $K_{q,\alpha} = 4^q \left[1 - 2^{-q(h-1/p)+(1-q/p)\alpha} \right]^{-1}$. Since $\{T_{2(k-1)} \leq T, T_{2k-1} = \infty\}$,

$k \geq 1$, are mutually disjoint, we have

$$(b-a) \sum_{k=1}^{\infty} \frac{1}{k^{1-\delta}} \mathbb{P}(U_a^b(D) \geq k) \leq \mathbb{E} \left[\sup_{u,v \in [0,T]} |X_u - X_v| \right] + K_{q,\alpha} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^h \sum_{k=1}^{\infty} \frac{1}{k^{1-\delta}} \mathbb{E} \left(\left| \frac{T_{2k-1} \wedge T - T_{2(k-1)} \wedge T}{T} \right|^\alpha \right)^{1-q/p}. \quad (27)$$

Since $0 < \delta < 1 - \frac{1}{ph}$, one may choose $q < p$ and then α such that $\frac{\delta}{h-1/p} < q < 1/h$ and $\frac{\delta}{1-q/p} < \alpha < \frac{h-1/p}{1/q-1/p} < 1$. By (27) and Hölder's inequality,

$$(b-a) \sum_{k=1}^{\infty} \frac{1}{k^{1-\delta}} \mathbb{P}(U_a^b(D) \geq k) \leq \left[\mathbb{E} \left(\sup_{u,v \in [0,T]} |X_u - X_v|^p \right) \right]^{1/p} + K_{q,\alpha} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^h \sum_{k=1}^{\infty} \frac{1}{k^{1-\delta}} \mathbb{E} \left(\left| \frac{T_{2k-1} \wedge T - T_{2(k-1)} \wedge T}{T} \right| \right)^{\alpha(1-q/p)} \leq C_{p,h,\theta}^{1/p} A_{p,h}^{1/p} T^h + K_{q,\alpha} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^h \left(\sum_{k=1}^{\infty} k^{-\frac{1-\delta}{1-\alpha(1-q/p)}} \right)^{1-\alpha(1-q/p)} \times \left[\sum_{k=1}^{\infty} \mathbb{E} \left(\left| \frac{T_{2k-1} \wedge T - T_{2(k-1)} \wedge T}{T} \right| \right) \right]^{\alpha(1-q/p)}. \quad (28)$$

It follows from $\alpha > \frac{\delta}{1-q/p}$ that $\frac{1-\delta}{1-\alpha(1-q/p)} > 1$. Therefore, $\zeta \left(\frac{1-\delta}{1-\alpha(1-q/p)} \right) = \sum_{k=1}^{\infty} k^{-\frac{1-\delta}{1-\alpha(1-q/p)}} < \infty$. Note that the sequence $T_j \wedge T$, $j \geq 0$, is increasing and bounded by T . We deduce that

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E} \left(\left| \frac{T_{2k-1} \wedge T - T_{2(k-1)} \wedge T}{T} \right| \right) &\leq \sum_{k=1}^{\infty} \mathbb{E} \left(\frac{T_{2k} \wedge T - T_{2(k-1)} \wedge T}{T} \right) \\ &= \mathbb{E} \left(\lim_{k \rightarrow \infty} \frac{T_{2k} \wedge T - T_0 \wedge T}{T} \right) \\ &\leq 1. \end{aligned}$$

Now (28) and the above yield that

$$(b-a) \sum_{k=1}^{\infty} \frac{1}{k^{1-\delta}} \mathbb{P}(U_a^b(D) \geq k) \leq C_{p,h,\theta}^{1/p} A_{p,h}^{1/p} T^h + K_{q,\alpha} \zeta \left(\frac{1-\delta}{1-\alpha(1-q/p)} \right)^{1-\alpha(1-q/p)} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} T^h. \quad (29)$$

Since

$$(k+1)^\delta - k^\delta = k^\delta \left[\left(1 + \frac{1}{k} \right)^\delta - 1 \right] < k^\delta \cdot \frac{1}{k} = \frac{1}{k^{1-\delta}} \leq \frac{2}{(k+1)^{1-\delta}},$$

for any $0 < \delta < 1$, summing by parts, we deduce from (29) that

$$\begin{aligned}
\mathbb{E} (U_a^b(D)^\delta) &= \sum_{k=1}^{\infty} k^\delta \mathbb{P} (U_a^b(D) = k) \\
&= \sum_{k=1}^{\infty} (k^\delta - (k-1)^\delta) \mathbb{P} (U_a^b(D) \geq k) \\
&\leq 2 \sum_{k=1}^{\infty} \frac{1}{k^{1-\delta}} \mathbb{P} (U_a^b(D) \geq k) \\
&\leq \frac{2}{b-a} \left(C_{p,h,\theta}^{1/p} A_{p,h}^{1/p} + K_{q,\alpha} \zeta \left(\frac{1-\delta}{1-\alpha(1-q/p)} \right)^{1-\alpha(1-q/p)} C_{p,h,\theta}^{q/p} A_{p,h}^{q/p} \right) T^h.
\end{aligned}$$

This completes the proof.

By Fatou's lemma and Proposition 3.1, one has the following □

Theorem 3.1. *Let U_a^b be the up-crossing number of X_t through $[a, b]$. Then, for any $0 < \delta < 1 - \frac{1}{ph}$,*

$$\mathbb{E} \left[(U_a^b)^\delta \right] < \frac{K_\delta}{b-a} T^h,$$

where the constant K_δ is the same as in Proposition 3.1. In particular, $U_a^b < \infty$ a.s.

References

- [1] R. Paley and A. Zygmund, On some series of functions (1), In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 26, pages 337-357. Cambridge University Press, 1930.